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**LETTER TO THE EDITOR**

# Correlation timescales in the Sherrington–Kirkpatrick model

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**Abstract**
We investigate the dynamical behaviour of the Sherrington–Kirkpatrick mean field model of spin glasses by numerical simulation. All the timescales $\tau_x$ we have measured behave as $\ln(\tau_x) \propto N^{\epsilon}$, where $N$ is the number of spins and $\epsilon \simeq \frac{1}{3}$. This is true whether the autocorrelation function used to define $\tau_x$ is sensitive to the full reversal of the system or not.

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Today many features of the Sherrington–Kirkpatrick mean field model of spin glasses [1–3] have been clarified. Most of the questions that still need investigating are related to the very interesting dynamics of the model (see for example [4]). Here, following Mackenzie and Young [5], we examine the equilibrium dynamics of the model. In this classic paper the authors gave numerical evidence, from systems with up to 192 spins, for the existence of a spectrum of relaxation times which diverge with the number of spins $N$ as $\ln(\tau) \propto N^{1/4}$, and of a second, longer 'ergodic' timescale $\tau_{eg}$ which is the time needed to turn over all the spins, with $\overline{\langle \ln(\tau_{eg}) \rangle} \propto N^{1/2}$. In order to do this one looks both at the processes that require a full reversal of all the spins and at processes that, in contrast, are not sensitive to this phenomenon. In this letter, we establish that indeed all dynamical scales have the same behaviour, compatible with barrier heights growing as $N^{\epsilon}$, where $\epsilon \simeq 0.3$ is close to the $N^{1/3}$ behaviour suggested in [6–8] (see also the numerical simulations in [9]). This result is at variance with the findings of [5], where smaller lattices were used and different observables were analysed: the exponents determined in [5] can be regarded as effective exponents that govern the dynamics in a transient regime.

Let us start by giving some details about our simulation. We study systems with $N = 64$, 128, 256, 512 and 1024 spins, with $\pm 1$ couplings. We first thermalize the system using the *parallel tempering* optimized Monte Carlo procedure [10] with a set of 38 $T$ values in the range 0.4–1.325 (i.e. $\Delta T = 0.025$). We perform 400 000 iterations (one iteration consists of one Metropolis sweep plus one tempering update cycle), and store the final well equilibrated configurations. Next we start updating these equilibrium configurations (more precisely the subset with $T = 0.4, 0.5, \ldots$) with a simple Metropolis dynamics, and perform $4 \times 10^6$ Metropolis sweeps. We have in all cases two replica and 512 realizations of the disorder.

For each of these samples we compute the *flip times* $\tau_1$, $\tau_2$ and $\tau_3$. We define $\tau_1^{(J)}$ as the time after which, on a given sample $J$, the time-dependent self-overlap

$$q(0,t) \equiv \frac{1}{N}\sum_i \sigma_i(0)\sigma_i(t) \tag{1}$$

has become smaller than $+\Sigma$, with

$$\Sigma \equiv \sqrt{\langle q^2\rangle_J} \tag{2}$$

where $\langle q^2\rangle_J$, the usual square Parisi overlap, is computed during the second half of the parallel tempering run for the given sample. The time $t$ is measured in units of sweeps, with $t = 0$ at the beginning of the Metropolis dynamics. We define analogously $\tau_2$ as the time it takes to $q(0,t)$ to decay from its initial value of 1 down to 0, and $\tau_3$ as the time it takes to $q(0,t)$ to decay down to $-\Sigma$.

We expect[3] $\tau_1$, $\tau_2$ and $\tau_3$ to obey the same scaling law. In the following we will try to check if an exponential scaling of the kind

$$\tau_{1,2,3} \simeq A_{1,2,3}\exp\left(\alpha_{1,2,3}N^{\epsilon}\right) \tag{3}$$

gives a good fit to the data, and we will try to determine $\epsilon$.

We base our analysis on empirical *medians* for $\ln(\tau)$, i.e. we sort the 512 values of $\ln(\tau)$ as $\ln(\tau^{(0)}) \leqslant \ln(\tau^{(1)}) \leqslant \cdots \leqslant \ln(\tau^{(511)})$ (more precisely the 512 values of $\ln(\tau)$ averaged over the two replica) and define the median as $\ln(\tau^{(255)})$. For large $N$ and small $T$, the probability distribution of $\tau$ has a very long tail (for large values of $\tau$), and in many cases we are not able to compute average values, since for some samples $\tau$ is larger than the number of sweeps performed. In contrast, the median approach works, and allows a fair estimate. In all cases where we are also able to estimate the average value of $\ln(\tau)$, we find that it is very similar to the median value. Thanks to this approach we have been able to estimate $\tau_1$ on all our lattice sizes down to $T = 0.4$, $\tau_2$ down to $T = 0.5$ and $\tau_3$ down to $T = 0.6$. Statistical errors have been computed using the usual bootstrap procedure.

The second decay time of interest is the timescale that governs the decay of, for example, the square (time-dependent) overlap. We monitor the decay of $\overline{\langle q(0,t)\rangle_J}$ and of

$$q_c^2(t) \equiv \overline{\langle q^2(0,t)\rangle_J - \langle q^2\rangle_J} \tag{4}$$

and we call $\tau_q$ and $\tau_{q_2}$ the timescales that characterize the short-time decay of these objects (see later for details regarding the exact definition).

Let us start with the results for $\tau_1$, $\tau_2$ and $\tau_3$. In figure 1 we plot one of our most successful fits of $\tau_3$: here we are at $T = 0.6$, the fit is very good and we estimate

$$\epsilon_{\tau_3}(T = 0.6) = 0.25 \pm 0.04 \tag{5}$$

that can be compared to

$$\epsilon_{\tau_1}(T = 0.6) = 0.20 \pm 0.16 \qquad \epsilon_{\tau_2}(T = 0.6) = 0.19 \pm 0.07 \tag{6}$$

(the fits for the three different $\tau_i$, $i = 1, 2, 3$ are all good, of the same quality as the one we have shown in the figure). Our estimates for $\epsilon_{\tau_1}$, $\epsilon_{\tau_2}$ and $\epsilon_{\tau_3}$ turn out to be very similar. The general pattern that emerges from these fit is of a very good consistency. Let us go into some

[3] Notice that while $\tau_2$ and $\tau_3$ are unambiguous signatures of the transition to the reversed part of the phase space, $\tau_1$ can be ambiguous, since depending on $T$ it can still characterize a transition in the short-time regime or already an ergodic transition. The fact that the three $\tau_i$ turn out to be compatible gives further support to the existence of a single timescale exponent $\epsilon$.

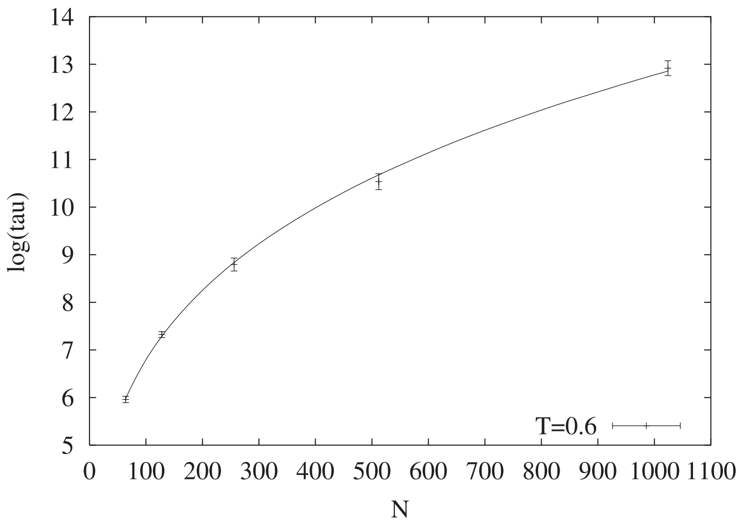


**Figure 1.** Points with errors are for $\ln(\tau_3)$ versus $N$, and the continuous curve for our best fit to the form (3).

more detail. Fits to $\epsilon_{\tau_3}$ (here, as we said, we wait for $q$ to become negative and equal to $-\Sigma$) are available only down to $T = 0.6$ (at lower $T$ values $\tau_3$ is too large and we are not able to estimate at all $\epsilon_{\tau_3}$). For $T$ from 0.6 up to 0.8 the best fit is very stable with an exponent close to 0.25–0.30. When going too close to the critical point the behaviour becomes less clean. $\epsilon_{\tau_2}$ (where we wait for $q$ becoming zero) can be determined down to $T = 0.5$ ($\tau_2$ is smaller than $\tau_3$). Here fluctuations are slightly larger than in the former case, but again up to $T = 0.8$ the exponent fluctuates in the range 0.2–0.3. In the $\epsilon_{\tau_1}$ case (where we only wait for $q$ decreasing from 1 down to $+\Sigma$) we succeed in obtaining a good estimate down to $T = 0.4$. Again here, for example, we estimate $\epsilon_{\tau_1}(T = 0.4) \simeq 0.25$, and we get a quite stable fit in $T$. We remark that when $T$ approaches $T_c$ the estimates of $\epsilon_{\tau_{1,2,3}}$ have large errors: $\alpha$ becomes very small (one expects $\alpha \to 0$ for $T \to T_c$) and the leading $N^\epsilon$ behaviour cannot be distinguished, with the present range of system sizes, from sub-leading corrections. It is also important to note that our data fully confirms that different ways to estimate the correlation times (the 1, 2 and 3 $\tau$) lead to the same scaling behaviour, with a scaling exponent close to $\simeq$0.3.

Let us note here (this is the focal point of this letter, as we will discuss in greater detail in the following) that the result of equation (5) *does not manifest*, as opposite to the findings of [5], a scale of the order of $\exp(cN^{\frac{1}{2}})$. The scale we observe is governed by an exponent close to 0.3.

We discuss now the measurements of correlation times that do not involve the reversal of all the spins. As an example we plot in figure 2 $q_c^2(t)$ versus $\ln(t)$ at $T = 0.6$, and in figure 3 the same quantity at $T = 0.4$. The two figures exhibit two regimes separated by some crossover value $t_{\max}$: a small-time regime, where $q_c^2(t)$ decays slowly with $\ln(t)$, and a large-$t$ regime where $q_c^2(t)$ is very small. This is very suggestive of the existence of a whole spectrum of relaxation times, up to some maximal value $\approx t_{\max}$.

We have defined the correlation time $\tau_{q_2}$ by computing the time needed for $q_c^2(t)$ to decrease from the value 0.25 to a threshold value $\mathcal{T}$ that we vary ([5] was looking directly to the moment

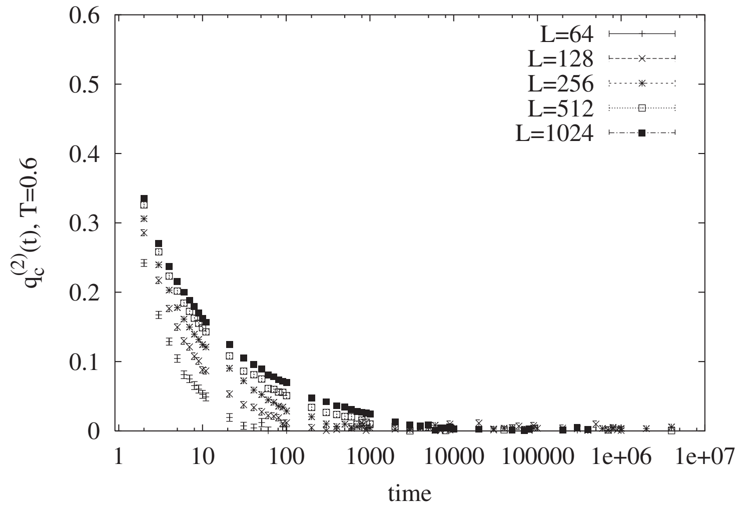


**Figure 2.** $q_c^2(t)$ versus $\ln(t)$ at $T = 0.6$.

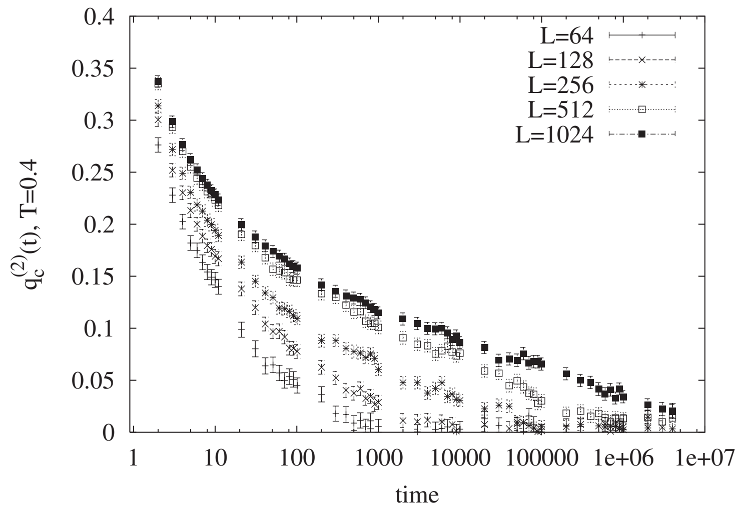


**Figure 3.** $q_c^2(t)$ versus $\ln(t)$ at $T = 0.4$.

in which $q_c^2(t)$ is close enough to zero)[4]. In the case of $q_c^2(t)$ we have used the two threshold values $\mathcal{T}_1 = 0.125$ and $\mathcal{T}_2 = 0.050$.

The exponents we estimate by best fits to the form (3) are again quite stable (even if in this case we have not been able to produce reliable error estimates) and, let us note right away, if any they are larger than the one estimated for the full reversal times $\tau_{1,2,3}$: we can be quite precise in the claim that the scenario where a slower timescale governs the full spin reversal while a faster timescale governs the valley-to-valley migration does not apply. As an example, we plot in figure 4 the $\tau_{q_2}$ time as a function of $N$, and our best fit to the form (3)

[4] It is important to note that the ergodic correlation times $\tau_i$ and these $\tau_{q_2}$, $\tau_q$ are defined in very different ways, and none of them as a simple, *bona fide* coefficient of an exponential decay $e^{-t/\tau}$. The fact that we find that they satisfy reasonable scaling laws shows that the definitions we use are well founded.

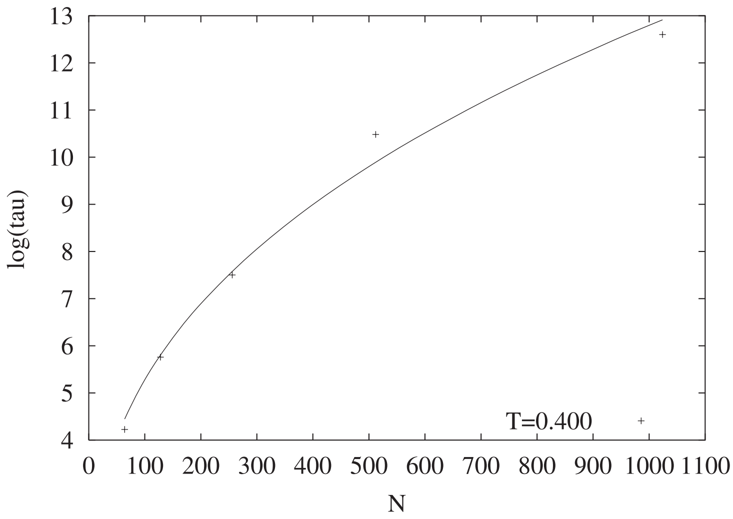


**Figure 4.** Data points are for $\ln(\tau_{q_2})$ (without error-bars) versus $N$, and the continuous curve for our best fit to the form (3).

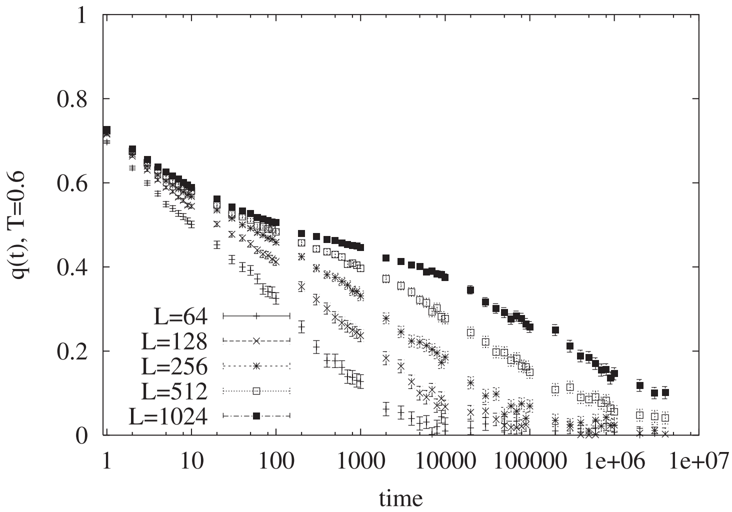


**Figure 5.** $q(t)$ versus $\ln(t)$ at $T = 0.6$.

at $T = 0.4$ and for a threshold $\mathcal{T}_2 = 0.050$: the estimated exponent here is $0.38 \pm 0.05$. The exponent values are very stable when changing the value of the lower threshold, which is a very good sign. In the $T$ range 0.5–0.8 the estimated value of $\epsilon$ lies in the range 0.28–0.38, i.e. completely compatible with the value $\frac{1}{3}$ that is reasonable from a theoretical point of view (see, for example, [6, 7]). The quality of the best fit degenerates again when $T$ becomes too close to $T_c$. It is may be worth stressing here that the determination of the exponent $\epsilon$ is a very difficult problem, exponentially more difficult than the usual determination of critical exponents, since here instead of a power behaviour we are trying to fit an exponential to a power behaviour: if $\tau$ ranges over five orders of magnitude (which would be more than acceptable for a power fit) its

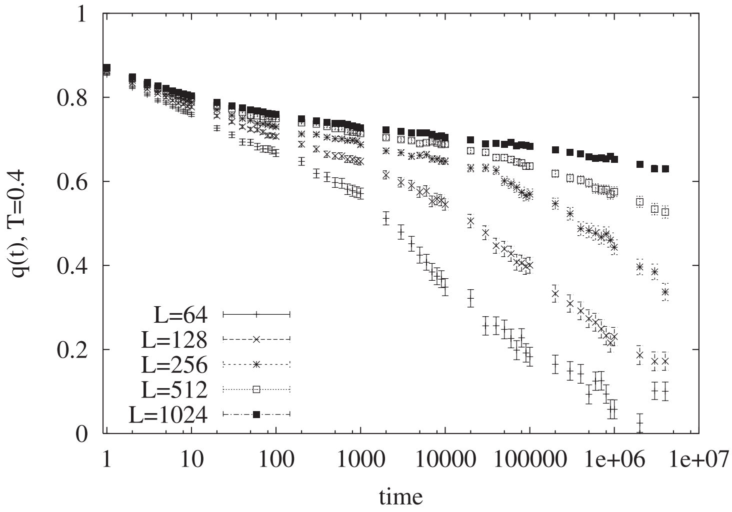


**Figure 6.** $q(t)$ versus $\ln(t)$ at $T = 0.4$.

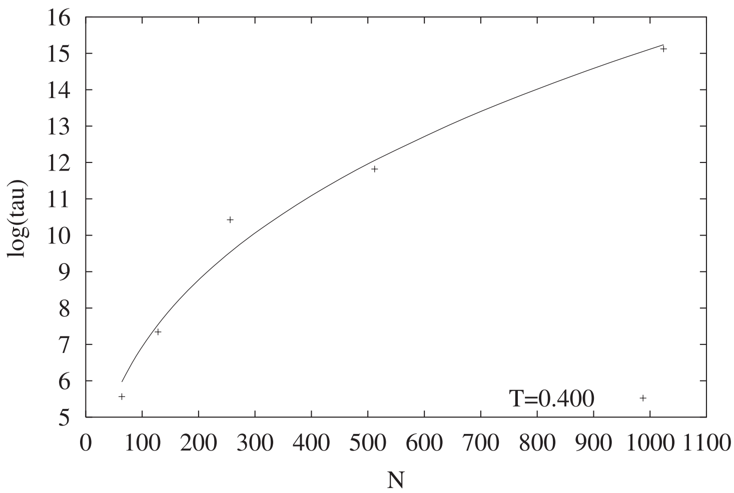


**Figure 7.** Data points are for $\ln(\tau_q)$ (without error-bars) versus $N$, and the continuous curve for our best fit to the form (3).

logarithm ranges over half a decade only, which gives a poor basis for our fit to the exponential of a power law.

We have also measured $q(t)$, which we plot in figures 5 and 6 for $T = 0.6$ and 0.4 respectively. At large times $q(t)$ goes to zero, in contrast we expect the initial decay to be governed from the same process that determines the decay of $q_c^2(t)$. It is also interesting to note that we are observing the expected plateau at the Edwards–Anderson value of the self-overlap, $q_{EA}$: with good approximation one estimates [2] $q_{EA}(T = 0.6) \simeq 0.50$ and $q_{EA}(T = 0.4) \simeq 0.74$. These two values coincide very well with the locations where on our larger lattice we see a plateau: this is very clear at $T = 0.4$ in figure 6 and a bit less clean but also evident at $T = 0.6$ in figure 5. The finite, large system, spends a long time at $q_{EA}$ before having $q(t) \to 0$ because of the ergodic transition.

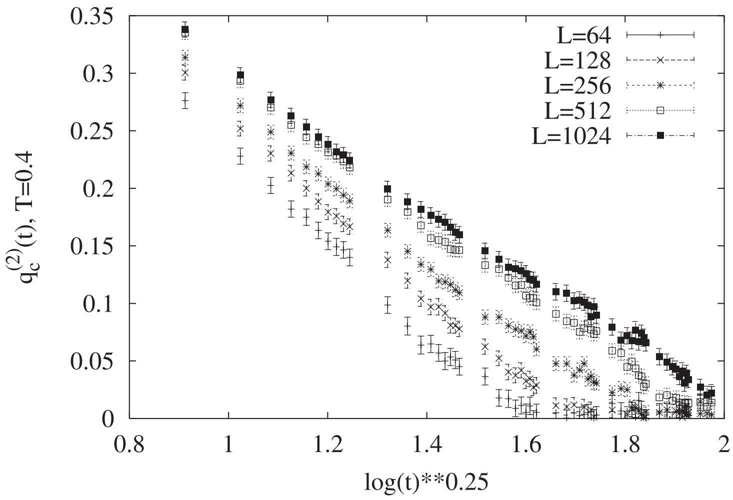


**Figure 8.** $q_c^2(t)$ versus $\ln(t)^{0.25}$ at $T = 0.4$.

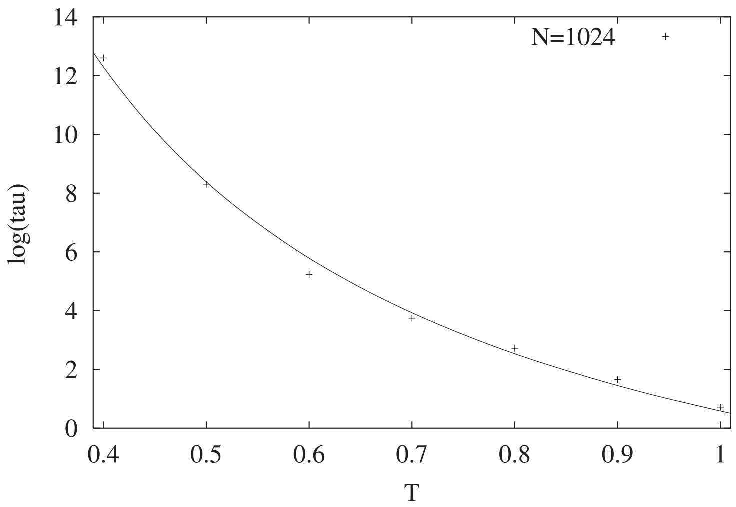


**Figure 9.** $\ln(\tau_{q_2})$ versus $T$ for $N = 1024$. The solid curve is our best fit to an Arrhenius behaviour.

We have checked that by fitting with the same procedure used for $q_c^2(t)$, using this time the $q$ interval going from 1 down to 0.63 (we use a higher low threshold to stay far from the actual decay to zero). Things work well, and we fit a scaling exponent for the correlation times statistically compatible with the one obtained for $q_c^2(t)$. We show in figure 7 the analogue of figure 4, where the best fit gives $\epsilon_{\tau_q} = 0.34 \pm 0.02$ (again, highly compatible with the value $\frac{1}{3}$). Consistent results (slightly lower, of the order of 0.25) are obtained at higher $T$ values.

It is clear from the figures we have shown that $q(t)$ and $q_c^2(t)$ decay very slowly to zero on a logarithmic scale. We try to be more quantitative in figure 8, where we show that the $q_c^2(t)$ data are very linear when plotted, for example, as a function of $\ln(t)^{\beta}$, with $\beta = 0.25$: we do not consider that as a fair determination of $\beta$, since there is a large range of values of $\beta$

which makes the plot linear. What we can claim is that $\beta$ is surely a small value, of the order of magnitude of 0.25. At higher $T$ values we have the same kind of behaviour.

In figure 9 we show, for our largest system, $\ln(\tau_{q_2})$ as function of $T$. The data are very well explained by the fact that we expect an Arrhenius-like behaviour, $\exp\left(\frac{A}{T}\right)$, with $A \simeq (T_c - T)$ [6]: a coefficient proportional to $\frac{T_c-T}{T}$ fits the data very well indeed.

We can sketch a few conclusions. In the Sherrington–Kirkpatrick mean field model of spin glasses one single time scaling dictates the behaviour of the correlation times related to the complete reversal of all spins and to the transitions through the different states that constitute the phase space: the speculation suggesting that one could get two different scaling laws is unfounded. It is not easy to get precise values for the exponent that characterizes this exponential scaling, but all our findings are compatible with a $\epsilon = \frac{1}{3}$ scaling: this is consistent with barrier scaling such as $N^{1/3}$ [6, 7]. We have also been able to show that the connected squared overlap decays to zero with a power of the logarithm of the order of 0.25 (and clearly not like a power law).

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